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Classical Linear Connections from Projectable Ones on Vertical Weil Bundles

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Abstract. We essentially reduce the problem of describing all natural operators $Q_{proj} \rightsquigarrow QV^A$ lifting projectable classical linear connections on fibred manifolds to classical linear connections on vertical Weil bundles.

1.

All manifolds are assumed to be without boundaries, finite dimensional, Hausdorff, second countable and smooth (of class C^∞) and maps to be of class C^∞. Unless otherwise specified, we use the terminology from [3]. In particular, $\mathcal{FM}_{m,n}$ denotes the category of all fibred manifolds with n-dimensional fibres and m-dimensional bases and their (fibred) embeddings.

In [4], given a Weil algebra A, $k+1 = dim_{\mathbf{R}} A$, we reduced the problem of finding of all natural operators $\Lambda : Q \rightsquigarrow QT^A$ lifting classical linear connections to the Weil functor T^A to the one of finding of all natural operators $C : Q \rightsquigarrow (T \times \ldots \times T, T^* \otimes T^* \otimes T)$ sending classical linear connections $\nabla \in Con(M)$ into fibred maps $C_M(\nabla) : TM \times_M \ldots \times_M TM(k \text{ times of } TM) \to T^*M \otimes T^*M \otimes TM$. In the present note we study the same problem for T^A replaced by the vertical Weil functor V^A. More precisely, we reduce the problem of finding of all natural operators $\Lambda : Q_{proj} \rightsquigarrow QV^A$ lifting projectable classical linear connections to V^A to the one of finding of all natural operators $C : Q_{proj} \rightsquigarrow (V \times \ldots \times V, F_1^* \otimes F_2^* \otimes F_3)$ sending projectable classical linear connections $\nabla \in Con_{proj}(Y)$ on fibred manifolds Y into fibred maps $C_Y(\nabla) : VY \times_Y \ldots \times_Y VY(k \text{ times of } VY) \to (F_1Y)^* \otimes (F_2Y)^* \otimes F_3Y$ covering id_Y, where F_1, F_2, F_3 are T or V, and where T is the tangent functor and V is the vertical functor.

2.

A Weil algebra A is a finite dimensional real local commutative algebra with unity 1_A (i.e. $A = \mathbf{R}.1_A \oplus N_A$, where N_A is an ideal of nilpotent elements). In [6], A. Weil introduced the concept of near A-points on a manifold M as an algebra homomorphisms of the algebra $C^\infty(M)$ of smooth real valued functions on M into a Weil algebra A. The space T^A of all near A-points on M is called a Weil bundle. The functor T^A sending

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any manifold M into T^AM and any map $f : M \to N$ into $T^Af : T^AM \to T^AN$ is called the Weil functor corresponding to A.

Given a Weil algebra A, the vertical Weil functor V^A sends any fibred manifold $Y \to M$ into the bundle $V^AY := \bigcup_{x\in M} T^AY_x$ over Y and any fibred map $f : Y \to Y^1$ into $V^Af := \bigcup_{x\in M} T^Af_x : V^AY \to V^AY^1$. If $A = \mathbf{D}$ is the Weil algebra of dual numbers, then $T^{\mathbf{D}} = T$ and $V^{\mathbf{D}} = V$ is the (usual) vertical functor.

3.

A linear connection in a vector bundle E over N is a bilinear map $D : \mathcal{X}(N) \times \Gamma E \to \Gamma E$ such that

$$D_{fX}\sigma = fD_X\sigma \text{ and } D_X f\sigma = Xf\sigma + fD_X\sigma$$

for any $X \in \mathcal{X}(N)$, $f \in C^\infty(N)$ and $\sigma \in \Gamma E$, where $C^\infty(N)$ is the algebra of C^∞-maps $N \to \mathbf{R}$, $\mathcal{X}(N)$ is the $C^\infty(N)$-module of vector fields on N and ΓE is the $C^\infty(N)$-module of smooth sections of E.

In particular, a linear connection ∇ in the tangent space TN of a manifold N is a classical linear connection on N.

It is well known (see [2]), that if ∇ is a classical linear connection on a manifold N and $x \in N$ then there is a so called ∇-normal coordinate system $\varphi : (N, x) \to (\mathbf{R}^n, 0)$ with center x. If $\psi : (N, x) \to (\mathbf{R}^n, 0)$ is another ∇-normal coordinate system with center x then there is $B \in GL(n)$ with $\psi = B \circ \varphi$ near x.

A classical linear connection ∇ on the total space Y of a fibred manifold $p : Y \to M$ is projectable if there is a (unique) classical linear connection $\underline{\nabla}$ on M such that ∇ and $\underline{\nabla}$ are p-related.

If ∇ is a projectable classical linear connection on a fibred manifold Y over M being p-related to $\underline{\nabla}$, then there is a ∇-normal fibred coordinate system φ with center $y \in Y$ (covering a $\underline{\nabla}$-normal one with center $p(y)$). If ψ is another such ∇-normal fibred coordinate system with (the same) center y then $\psi = B \circ \varphi$ for some fibred linear isomorphism B.

4.

We have the following important proposition.

Proposition 1. *Let $A = \mathbf{R}.1_A \oplus N_A$ be a Weil algebra with $k + 1 = dim_{\mathbf{R}}A$ and ∇ be a projectable classical linear connection on a fibred manifold Y. There is a (canonical in ∇) fibred diffeomorphism*

$$I_\nabla : V^AY \to VY \otimes N_A = VY \times_Y \ldots \times_Y VY \; (k \text{ times of } VY)$$

covering the identity map of Y.

Proof. Let $v \in V^A_yY$, $y \in Y$. Let $\varphi : (Y, y) \to (\mathbf{R}^{m,n}, (0, 0))$ be a ∇-normal fibred coordinate system on Y with center y, where $\mathbf{R}^{m,n}$ is the trivial bundle over $\mathbf{R}^m$ with fiber $\mathbf{R}^n$. We put

$$I_\nabla(v) = I^\varphi_\nabla(v) = (V\varphi^{-1} \otimes id_A) \circ I \circ V^A\varphi(v) \in V_yY \otimes N_A \, ,$$

where $I : V^A_{(0,0)}(\mathbf{R}^{m,n}) = T^A_0\mathbf{R}^n = \mathbf{R}^n \otimes N_A \to V_{(0,0)}(\mathbf{R}^{m,n}) \otimes N_A = T_0\mathbf{R}^n \otimes N_A = \mathbf{R}^n \otimes N_A$ is the obvious $GL(m, n)$-invariant identification (the "identity" map). Clearly, $GL(m, n)$ is the space of fibred linear isomorphisms $\mathbf{R}^{m,n} \to \mathbf{R}^{m,n}$. If $\psi : (Y, y) \to (\mathbf{R}^{m,n}, (0, 0))$ is another ∇-normal fibred coordinate system on Y with center y, then $\psi = B \circ \varphi$ (near y) for some $B \in GL(m, n)$. Using the $GL(m, n)$-invariance of I we easily see that $I^\varphi_\nabla(v) = I^\psi_\nabla(v)$. So, $I_\nabla(v)$ is independent of the choice of φ.

5.

In [1], J. Gancarzewicz presented a canonical construction of a classical linear connection on the total space of a vector bundle E over N from a linear connection D in E by means of a classical linear connection ∇ on N. More precisely, if X is a vector field on N and σ is a section of E, then $D_X\sigma$ is a section of E. Further, let X^D denote the horizontal lift of a vector field X with respect to D. Moreover, using the translations in the individual fibers of E, we derive from every section $\sigma : N \to E$ a vertical vector field σ^V on E called the vertical lift of σ. In [1] J. Gancarzewicz proved the following fact.

Proposition 2. *For every linear connection D in a vector bundle E over N and every classical linear connection ∇ on N there exists a unique classical linear connection $\Theta = \Theta(D, \nabla)$ on the total space E with the following properties*

$$\Theta_{X^D} Y^D = (\nabla_X Y)^D,\ \Theta_{X^D}\sigma^V = (D_X\sigma)^V\ ,$$

$$\Theta_{\sigma^V} X^D = 0,\ \Theta_{\sigma^V}\sigma_1^V = 0$$

for all vector fields X, Y on N and all sections σ, σ_1 of E.

6.

The general concept of natural operators one can find in [3]. In the present note we use the following particular cases of natural operators.

An $\mathcal{FM}_{m,n}$-natural operator $\Lambda : Q_{proj} \rightsquigarrow QV^A$ (lifting projectable classical linear connections to V^A) is an $\mathcal{FM}_{m,n}$-invariant family $\Lambda = \{\Lambda_Y\}$ of regular operators

$$\Lambda_Y : Con_{proj}(Y) \to Con(V^A Y)$$

for any $\mathcal{FM}_{m,n}$-object Y, where $Con_{proj}(Y)$ is the space of all projectable classical linear connections on Y and $Con(V^A Y)$ is the space of all classical linear connections on $V^A Y$. The invariance means that if $\nabla \in Con_{proj}(Y)$ and $\nabla_1 \in Con_{proj}(Y^1)$ are φ-related for an $\mathcal{FM}_{m,n}$-map $\varphi : Y \to Y^1$ then $\Lambda(\nabla)$ and $\Lambda(\nabla^1)$ are $V^A\varphi$-related. The regularity means that Λ_Y transforms smoothly parametrized families of projectable classical linear connections on Y into smoothly parametrized families of classical linear connections on $V^A Y$.

For example, if ∇ is a projectable classical linear connection on a fibred manifold Y over M, we have a linear connection D^∇ in the vector space $VY \times_Y \ldots \times_Y VY$ (k times) over Y (where $k + 1 = dim_{\mathbf{R}} A$) given by

$$D^\nabla_X(\sigma_1, \ldots, \sigma_k) := (\nabla_X\sigma_1, \ldots, \nabla_X\sigma_k)\ ,$$

$\sigma = (\sigma_1, \ldots, \sigma_k) \in \Gamma(VY \times_Y \ldots \times_Y VY)$, $X \in \mathcal{X}(Y)$, and then (because of Propositions 1 and 2) we have the classical linear connection

$$\nabla^{V^A} := (I_\nabla^{-1})_*\Theta(D^\nabla, \nabla)$$

on $V^A Y$. We call ∇^{V^A} the horizontal lift of ∇ to $V^A Y$. The construction of ∇^{V^A} is an $\mathcal{FM}_{m,n}$-natural operator $\Lambda^{V^A} : Q_{proj} \rightsquigarrow QV^A$. Actually, $\Lambda^{V^A}_Y(\nabla) = \nabla^{V^A}$ for any $\nabla \in Con_{proj}(Y)$.

Another example of $\mathcal{FM}_{m,n}$-natural operator $\Lambda : Q_{proj} \rightsquigarrow QV^A$ (without using Proposition 2) one can find in [5].

Quite similarly, an $\mathcal{FM}_{m,n}$ natural operator $C : Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes T)$ is an $\mathcal{FM}_{m,n}$-invariant system $C = \{C_Y\}$ of regular operators

$$C_Y : Con_{proj}(Y) \to C^\infty_Y(VY \times_Y \ldots \times_Y VY, T^*Y \otimes T^*Y \otimes TY)$$

for any $\mathcal{FM}_{m,n}$-object Y, where $C^\infty_Y(VY \times_Y \ldots \times_Y VY, T^*Y \otimes T^*Y \otimes TY)$ is the space of all fibred maps $VY \times_Y \ldots \times_Y VY \to T^*Y \otimes T^*Y \otimes TY$ covering id_Y. Now, the invariance means that if $\nabla \in Con_{proj}(Y)$ and $\nabla^1 \in Con_{proj}(Y^1)$ are φ-related for some $\mathcal{FM}_{m,n}$-map $\varphi : Y \to Y^1$ then $C_Y(\nabla)$ and $C_{Y^1}(\nabla^1)$ are $(V\varphi \times \ldots \times V\varphi, T^*\varphi \otimes T^*\varphi \otimes T\varphi)$-related, i.e. $C_{Y^1}(\nabla^1) \circ (V\varphi \times \ldots \times V\varphi) = (T^*\varphi \otimes T^*\varphi \otimes T\varphi) \circ C_Y(\nabla)$. Replacing (in respective place) functor

T by functor V we obtain the concepts of $\mathcal{FM}_{m,n}$-natural operators $Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes V^* \otimes V)$, $Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes V^* \otimes T)$, e.t.c.

Many $\mathcal{FM}_{m,n}$-natural operators $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes T)$ or $Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes T^* \otimes T)$ or $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes V^* \otimes T)$ or $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes V^*)$ one can produce from the (higher order) covariant derivatives of the curvature tensor R^∇ or the torsion tensor T^∇ of $\nabla \in Con_{proj}(Y)$ by taking respective tensor product, contractions, symmetrization and linear combinations. For example we have the following $\mathcal{FM}_{m,n}$-natural operators in question.

— The family $C : Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes T)$ given by

$$C_Y(\nabla)(v_1, \ldots, v_k)(u, w) := \nabla^{k-1} R^\nabla(v_1, \ldots, v_k, u, w) \in T_y Y \tag{$*$}$$

for $v_1, \ldots, v_k \in V_y Y \subset T_y Y$, $u, w \in T_y Y$ and $y \in Y$ is an $\mathcal{FM}_{m,n}$-natural operator.

— An $\mathcal{FM}_{m,n}$-natural operator $Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes T^* \otimes T)$ is given by formula (*) for $v_1, \ldots, v_k \in V_y Y$, $u \in V_y Y$, $w \in T_y Y$ and $y \in Y$.

— An $\mathcal{FM}_{m,n}$-natural operator $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes V^* \otimes T)$ is defined by formula (*) for $v_1, \ldots, v_k \in V_y Y$, $u \in T_y Y$, $w \in V_y Y$ and $y \in Y$.

— An $\mathcal{FM}_{m,n}$-natural operator $C : Q_{proj} \rightsquigarrow (V \times V \times V, T^* \otimes T^* \otimes V)$ is given by

$$C_Y(\nabla)(v_1, v_2, v_3)(u, w) := Ric^\nabla(u, w) R^\nabla(v_1, v_2, v_3)\,,$$

$v_1, v_2, v_3 \in V_y Y$, $u, w \in T_y Y$, $y \in Y$, where Ric^∇ is the Ricci tensor field of ∇. (Indeed, if X_1, X_2, X_3 are vertical vector fields on Y, then $R^\nabla(X_1, X_2, X_3) = \nabla_{X_1}\nabla_{X_2} X_3 - \nabla_{X_2}\nabla_{X_1} X_3 - \nabla_{[X_1, X_2]} X_3$ is vertical because of $\nabla_U W$ is vertical for any projectable vector field U on Y and any vertical vector field W on Y (as ∇ is projectable).)

The problems of complete description of $\mathcal{FM}_{m,n}$-operators $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes T)$ and $Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes T^* \otimes T)$ and $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes V^* \otimes T)$ and $Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes V)$ are unsolved (and in our opinion rather unsolvable).

7.

Let $\Lambda : Q_{proj} \rightsquigarrow QV^A$ be an $\mathcal{FM}_{m,n}$-natural operator. Let $\nabla \in Con_{proj}(Y)$. We have the difference tensor field $\Delta^\Lambda(\nabla) := \Lambda_Y(\nabla) - \nabla^{V^A}$ of type $T^* \otimes T^* \otimes T$ on $V^A Y$. Applying $I_\nabla : V^A Y \to VY \times_Y \ldots \times_Y VY$ (from Proposition 1) we treat $\Delta^\Lambda(\nabla)$ as tensor field

$$\Delta^\Lambda(\nabla) = (I_\nabla)_*(\Lambda_Y(\nabla) - \nabla^{V^A})$$

of type $T^* \otimes T^* \otimes T$ on $VY \times_Y \ldots \times_Y VY$. On the other hand, using D^∇ (see above) we have the decomposition

$$T_v(VY \times_Y \ldots \times_Y VY) = V_v(VY \times_Y \ldots \times_Y VY) \oplus H_v^{D^\nabla}$$

for any $v \in (VY \times_Y \ldots \times_Y VY)_y$, $y \in Y$. Since $V_v(VY \times_Y \ldots \times_Y VY) = V_y Y \times \ldots \times V_y Y$ and $H_v^{D^\Lambda} = T_y Y$ (modulo the usual identifications), then

$$T_v(VY \times_Y \ldots \times_Y VY) = V_y Y \times \ldots \times V_y Y \times T_y Y$$

(modulo the identifications). Hence $\Delta^\Lambda(\nabla)(v) \in T_v^*(VY \times_Y \ldots \times_Y VY) \otimes T_v^*(VY \times_Y \ldots \times_Y VY) \otimes T_v(VY \times_Y \ldots \times_Y VY)$ can be treated as the system of $(k+1)^3$-tuples of k^3 elements from $(V_y Y)^* \otimes (V_y Y)^* \otimes V_y Y$ and k^2 elements from $(V_y Y)^* \otimes (V_y Y)^* \otimes T_y Y$ and ... and one element from $(T_y Y)^* \otimes (T_y Y)^* \otimes T_y Y$.

Thus we have proved the following reducibility theorem.

Theorem 1. *Let A be a Weil algebra, $k + 1 = dim_{\mathbf{R}} A$. The $\mathcal{FM}_{m,n}$-natural operators $\Lambda : Q_{proj} \rightsquigarrow QV^A$ lifting projectable classical linear connections to V^A are in bijection with the $(k+1)^3$-tuples $(C^I, C^{II}_a, C^{III}_b, \ldots, C^{VIII}_g)$ of $\mathcal{FM}_{m,n}$-natural operators*

$$C^I : Q_{proj} \rightsquigarrow (V \times \ldots \times V, T^* \otimes T^* \otimes T) \text{ and}$$

$$C^{II}_a : Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes V^* \otimes V) \text{ for } a = 1, \ldots, k^3 \ ,$$
$$C^{III}_b : Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes V^* \otimes T) \text{ for } b = 1, \ldots, k^2 \ ,$$
$$C^{IV}_c : Q_{proj} \rightsquigarrow (V \times \ldots \times V, V^* \otimes T^* \otimes V) \text{ for } c = 1, \ldots, k^2 \ ,$$
$$C^{V}_d : Q_{proj} \rightsquigarrow (V \times ... \times V, T^* \otimes V^* \otimes V) \text{ for } d = 1, \ldots, k^2 \ ,$$
$$C^{VI}_e : Q_{proj} \rightsquigarrow (V \times ... \times V, V^* \otimes T^* \otimes T) \text{ for } e = 1, \ldots, k \ ,$$
$$C^{VII}_f : Q_{proj} \rightsquigarrow (V \times ... \times V, T^* \otimes V^* \otimes T) \text{ for } f = 1, \ldots, k \ ,$$
$$C^{VIII}_g : Q_{proj} \rightsquigarrow (V \times ... \times V, T^* \otimes T^* \otimes V) \text{ for } g = 1, \ldots, k \ .$$

The above theorem reduces the problem of finding all $\mathcal{FM}_{m,n}$-natural operators $Q_{proj} \rightsquigarrow QV^A$ lifting projectable classical linear connections to V^A to the one of finding $\mathcal{FM}_{m,n}$-natural operators of some types being simpler than $Q_{proj} \rightsquigarrow QV^A$ and depending only on the dimension of A. This "reduction" shows that the problem of finding all $\mathcal{FM}_{m,n}$-natural operators $Q_{proj} \rightsquigarrow QV^A$ is rather unsolvable. On the other hand, this "reduction" brings a possibility to give many examples of $\mathcal{FM}_{m,n}$-natural operators $Q_{proj} \rightsquigarrow QV^A$.

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